Symmetries of Fano Varieties $/\mathbb{C}$

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Question: How large can $\mathrm{Aut}(X)$ be for X Fano?

C = sm. proj. curve$/\mathbb{C}$

(Fano) $\left.\begin{array}{l} g=0 \\ K_C<0 \end{array}\right\}$ $C \cong \mathbb{P}^1$, $\mathrm{Aut}(\mathbb{P}^1) \cong PGL_2(\mathbb{C})$

(CY) $\left.\begin{array}{l} g=1 \\ K_C=0 \end{array}\right\}$ $\mathrm{Aut}(C) \cong C(\mathbb{C}) \rtimes \mathrm{Aut}(C,0)$

$\mathbb{Z}/2$, $\mathbb{Z}/4$, or $\mathbb{Z}/6$

(canonically polarized) $\left.\begin{array}{l} g\geq 2 \\ K_C>0 \end{array}\right\}$ $|\mathrm{Aut}(C)| \leq 84(g-1)$

Def: A group G is Jordan if there exists $J := J(G)$ such that every finite subgroup of G has a normal abelian subgroup of index at most J.

Thm: (Jordan, 1878) $GL_N(\mathbb{C})$ is Jordan.

$J(GL_N(\mathbb{C})) = (N+1)!$ for $N \geq 71$.

(Collins, 2007)

$$S_{N+1} \hookrightarrow GL_N(\mathbb{C}) \quad \text{standard rep.}$$

<u>Thm</u>: (Prokhorov-Shramov, Birkar, '16)
Fix $n \geq 1$. Then exists J_n such that for any rationally connected variety X of dim. n, $Bir(X)$ is Jordan and

$$J(Bir(X)) \leq J_n.$$

$\Rightarrow$ same result for $Aut(X)$, X Fano

<u>Cor</u>: For any dim. n, there exists a $m := m(n)$ such that if S_k acts faithfully on X, X RC and n-dim., then $k \leq m(n)$.

However: no effective bounds known on $m(n)$, $n \geq 4$

<u>Ex</u>: $S_k \circlearrowright \mathbb{P}^n$. $\quad Aut(\mathbb{P}^n) = PGL_{n+1}(\mathbb{C})$

Then $\begin{cases} k \leq 4, & n = 1, 2 \\ k \leq 6, & n = 3 \\ k \leq n+2, & n \geq 4 \end{cases}$

Ex: Low Dimensions

- $m(1) = 4$, $S_4 \circlearrowright \mathbb{P}^1$

- $m(2) = 5$ (Dolgachev–Iskovskikh, '09)

Three minimal examples:

$\mathbb{P}^1 \times \mathbb{P}^1$, Clebsch cubic surface, del Pezzo of deg. 5

$\{\Sigma x_i = \Sigma x_i^2 = 0\} \subseteq \mathbb{P}^4$ $\{\Sigma x_i = \Sigma x_i^3 = 0\} \subseteq \mathbb{P}^4$ $\overline{M}_{0,5}$

- $m(3) = 7$ (Prokhorov, '22)

Only example (up to conjugation)

$$X = \{ \Sigma x_i = \Sigma x_i^2 = \Sigma x_i^3 = 0 \} \subseteq \mathbb{P}^6$$

Note: X is irrational (Beauville, '12)

Bounds on Symmetric Actions

Thm: (EJM) $m(n)$ = largest sym. group action on a RC var. of dim. n

For any $\epsilon > 0$, $m(n) < (1+\epsilon)(n+1)^2$ for $n \gg 0$.

Example: Proof of theorem shows

$$m(4) \leq 34$$
$$m(5) \leq 41$$
$$\vdots$$

Proof:

Idea: Fixed point existence results of Haution.

Thm: (Haution, '19) Let X be a proj. var. $/k = \bar{k}$ with an action of a p-group G, $\dim X < p-1$.
Then $X(k)^G = \emptyset$ iff $p \mid \chi(X, \mathcal{F})$ for all $\mathcal{F}$ G-equivariant coherent sheaf on X.

Thm: (J. Xu, '20) If $G \leq \mathrm{Aut}(X)$ is a finite p-group, X RC of dim. $n < p-1$, then G is abelian of rank at most n.

Pf: Equivariant res. $\Rightarrow$ can assume X smooth
Take $\mathcal{F} = \mathcal{O}_X$. X is RC $\Rightarrow \chi(\mathcal{O}_X) = 1$.
Any G-action has a fixed point

x. G acts faithfully on $T_{X,x} \cong \mathbb{C}^n$

$(H \subseteq G, \quad T_{X^H,x} = (T_{X,x})^H)$

By rep. theory, G nonabelian $\Rightarrow \dim T_{X,x} \geq p$, contradiction.

$\Rightarrow$ G abelian, rank $\leq n$. $\square$

Return to symmetric groups:

A Sylow p-subgroup of S_k is nonabelian or rank $\geq n+1$ if $k \geq (n+1)p$

prime larger than n

(assume $p > n+1$ for Haution's thm. to apply)

Conclusion: $M(n) \leq (n+1)p$

$p > n+1$ is any prime.

(can make $p < (n+1)(1+\epsilon)$ for any $\epsilon > 0$ when $n >> 0$). $\square$

Ex: $X = \{ \sum x_i = \sum x_i^2 = \cdots = \sum x_i^m = 0 \} \subseteq \mathbb{P}^{n+m}$

dim. n, smooth

Choose m largest possible with

$(1+\cdots+m)-(n+m+1)<0 \Rightarrow X$ Fano

Then X has a S_{n+m+1}

For X of dim. n, get

$$k = n+m+1 = n + \left\lceil \frac{1+\sqrt{8n+9}}{2} \right\rceil$$

as the max possible.

n	k
1	4
2	5
3	7
4	8
5	9
6	11

Thm: (EJM) Let $X \subseteq \mathbb{P}(a_0, \dots, a_N)$ be a quasismooth weighted complete intersection of dim. n, with a faithful S_k-action.

Then $k \leq n + \left\lceil \frac{1+\sqrt{8n+9}}{2} \right\rceil$. This bound is sharp.

Proof idea: (no linear equation)

1) X Fano $\Rightarrow \operatorname{codim}_{\mathbb{P}}(X) < \dim(X)$

ex) $(2, \dots, 2)$ comp int in $\mathbb{P}^N$ is Fano iff dim < codim

2) Lift S_k action to $\mathrm{Aut}(\mathbb{P})$

$$\mathrm{Aut}(\mathbb{P}) \supseteq \bigoplus GL_{N_i}(\mathbb{C}), \quad N_i = N+1$$

maximal reductive subgroup

3) Bound k using projective rep theory of S_k $\square$

<u>Ex</u>: $n = 5$, theorem says $k \leq 9$

$$X \in \mathbb{P}^9(\underset{x_i}{1^{(9)}}, \underset{Y}{2})$$

$$X := \{ \textstyle\sum x_i = \sum x_i^2 = \sum x_i^3 = Y^2 - \sum x_i^4 = 0 \} \circlearrowleft S_9$$

(double cover of prev. example in $\mathbb{P}^8$)

<u>Thm</u>: (EJM)

Let $S_k \circlearrowright X$ a simplicial toric var. of dim. n

n	max k	Optimal Example
1	4	$\mathbb{P}^1$
2	5	$\mathbb{P}^1 \times \mathbb{P}^1$
3	6	$\mathbb{P}^3$
4	6	$\mathbb{P}^4$, $\mathbb{P}^2 \times \mathbb{P}^2$
$n \geq 5$	$n+2$	$\mathbb{P}^n$

Symmetric Actions & Boundedness

klt Fano var.'s are unbounded in dim. ≥ 2

Question: How do you create bounded classes of Fanos?

- constrain sing.'s (ϵ-lc Fanos are bounded in every dim. n, Birkar)

- constrain α-inv., and/or $(-K_X)^n$ (Birkar, C. Jiang)

- large symmetric group action?

<u>Thm</u>: (EJM) S_k-equivariant klt Fano fourfolds are bounded when $k \geq 8$. In contrast, they are unbounded for $k \leq 7$.

Recall: $k=8$ should be the max. possible

only known examples: $(1,2,3)$- c.i. in $\mathbb{P}^7$

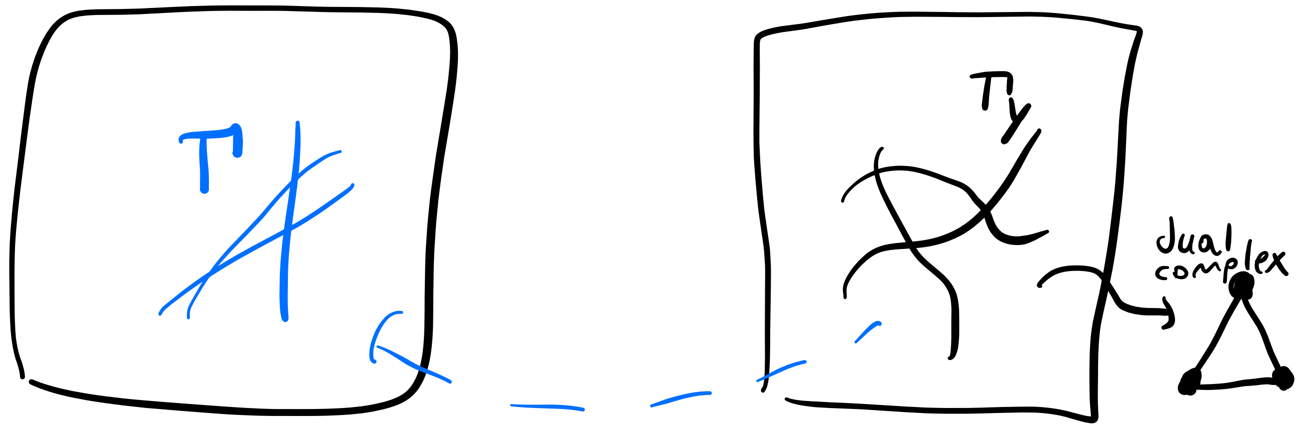
$(1,2,4)$-c.i. in $\mathbb{P}^7$

<u>Proof idea</u>:

pair w/ standard coeff.

Let $(Y, B_Y) := X/S_k$

Break into cases based on coreg(Y, B_Y)



$X \longrightarrow (Y, B_Y)$

Γ = S_k-equiv. boundary

$K_X + \Gamma \equiv 0$

$\dim \mathcal{D}(Y, \Gamma_Y) = \dim \mathcal{D}(X, \Gamma)$

$\Gamma_Y \geq B_Y$

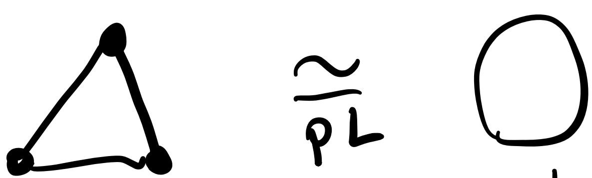
$K_Y + \Gamma_Y \equiv 0$,

make dual complex of $\Gamma_Y^{=1}$ as large as possible

Case 1: $\dim \mathcal{D}(X, \Gamma) \leq 0$

$\Rightarrow$ Prove boundedness directly

Case 2: $\dim \mathcal{D}(X, \Gamma) \geq 1$



$S_k \curvearrowright \mathcal{D}(X, \Gamma) \simeq_{PL}$ $\Rightarrow S_k \curvearrowright S^1$

in general, $\mathcal{D}(X, \Gamma)$ is the PL quotient of a sphere by a finite group for $\dim X \leq 4$

Either: • $G \twoheadrightarrow A_8$ ~~acts faithfully~~ of ~~sphere~~ of dim ≤ 3

impossible

- ~~vertex of dual complex is fixed $\Rightarrow$ (D, Γ_D) CY pair of dim. 3 with faithful G-action $G \twoheadrightarrow A_8$, $\Gamma_D \neq 0$~~

impossible.

$\square$